Last Time: Span + Lin. indep.

Claim: Given (Finite) $S \subseteq V$, there is a lin. indep subset $I \subseteq S$ w/ $\text{span}(I) = \text{span}(S)$.

Ex: Compute a subset I of $\left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\} = S$ w/ I indep and $\text{span}(I) = \text{span}(S)$.

Sol: $\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 1 \end{bmatrix} \vec{x} = \vec{b} \in \mathbb{R}^3$.

$$\to \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & 0 \end{array}\right]$$

No leading 1

$$\rightsquigarrow \begin{cases} c_1 + c_3 + \frac{1}{2}c_5 = 0 \\ c_2 - c_3 + \frac{1}{2}c_5 = 0 \\ c_4 + \frac{1}{2}c_5 = 0 \end{cases} \qquad \vec{x} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}$$

Use $I = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\}$ which is linearly independent because the corresponding columns of RREF(M) all have leading 1's. ☑

Bases and Dimension

Defn: Let V be a vector space. A *basis* of V is a linearly independent, spanning subset of V. ← *ordered...*

Ex: In $\mathbb{R}^2$, $B = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is a basis.

$B' = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ is a *different basis!*

We'll solve the linear system $\left[\begin{array}{cc|c} 3 & -1 & a \\ 1 & 1 & b \end{array}\right]$.

$$\left[\begin{array}{cc|c} 3 & -1 & a \\ 1 & 1 & b \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & b \\ 3 & -1 & a \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & b \\ 0 & -4 & a-3b \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & +\frac{1}{4}a + \frac{1}{4}b \\ 0 & 1 & -\frac{1}{4}(a-3b) \end{array}\right] \leftarrow$$

$$\therefore \begin{bmatrix} a \\ b \end{bmatrix} = \left(\tfrac{1}{4}a + \tfrac{1}{4}b\right) \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \left(\tfrac{1}{4}a - \tfrac{3}{4}b\right) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

recipe to construct

Note $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, we obtain unique solution

$$0 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

So B is lin. indep.

On the other hand, given $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ there are coefficients (namely $c_1 = \frac{1}{4}a + \frac{1}{4}b$ and $c_2 = -\frac{1}{4}a + \frac{3}{4}b$) for which $\begin{bmatrix} a \\ b \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$,

So $\begin{bmatrix} a \\ b \end{bmatrix} \in \operatorname{Span}\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$. Hence B is a basis. ☒

<u>Non-Ex</u>: $D = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix} \right\}$ is <u>Not</u> a basis of $\mathbb{R}^3$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 1 & 1 & 0 & b \\ 0 & 1 & -1 & c \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & -1 & -a+b \\ 0 & 1 & -1 & c \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & -1 & -a+b \\ 0 & 0 & 0 & a-b+c \end{array}\right]$$

So $\begin{bmatrix}a\\b\\c\end{bmatrix} \in \text{span}(D)$ implies $a-b+c=0$

Thus $\text{span}(D) \neq \mathbb{R}^3$ (right away: Not a basis).

Alternatively, $a=b=c=0$, then we have $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

So $\begin{cases} c_1 + c_3 = 0 \\ c_2 - c_3 = 0 \end{cases} \rightsquigarrow$ <u>$c_3 = -c_1 = c_2$</u>

$\therefore$ we have a nontrivial linear combination resulting in $\vec{0}$:

$$1\begin{bmatrix}1\\1\\0\end{bmatrix} - 1\begin{bmatrix}0\\1\\1\end{bmatrix} - 1\begin{bmatrix}1\\0\\-1\end{bmatrix} = \vec{0},$$ so D is lin. dep. ☑

<u>Ex</u>: ① Let $A = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\} \subseteq \mathbb{R}^3$.

$\text{span}(A) \neq \mathbb{R}^3$, but A is lin indep.

② Let $A' = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix} \right\} \subseteq \mathbb{R}^3$ has

$\text{span}(A') = \mathbb{R}^3$, but A is lin. dep.

<u>Defⁿ</u>: In $\mathbb{R}^n$, the <u>standard basis</u> is

$$\mathcal{E}_n = \{e_1, e_2, \dots, e_n\}$$

where $e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ ith position.

<u>Ex</u>: In $\mathbb{R}^2$, $\mathcal{E}_2 = \left\{ \underset{e_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \underset{e_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right\}$.

In $\mathbb{R}^3$, $\mathcal{E}_3 = \left\{ \underset{e_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{e_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{e_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\}$ etc.

<u>Q</u>: $\{0_V\} \subseteq V$ is the trivial subspace.
what is a basis for $\{0_V\}$?

<u>A</u>: $0_V \in \text{Span}(S)$ for all $S \subseteq V$.
$\therefore$ $0_V \in \text{span}(\emptyset)$. So $\emptyset$ spans $\{0_V\}$
and (from last time) $\emptyset$ is lin. indep,
so $\emptyset$ is a basis of $\{0_V\}$. ☑

<u>Ex</u>: $\mathcal{P}_3(\mathbb{R}) = \{$ polys of degree at most 3 $\}$.
$B = \{1, x, x^2, x^3\}$ is a basis.

$$a + bx + cx^2 + dx^3 = c_0 1 + c_1 x + c_2 x^2 + c_3 x^3$$

$\Rightarrow \begin{cases} c_0 = a \\ c_1 = b \\ c_2 = c \\ c_3 = d \end{cases}$ uniquely solvable for all $a, b, c, d \in \mathbb{R}$.
Hence B is a basis of $\mathcal{P}_3(\mathbb{R})$.

$B' = \{1+x,\ x+x^2,\ x^2+x^3,\ 1+x^3\}$ is a basis.

$$b_0 + b_1x + b_2x^2 + b_3x^3 = c_0(1+x) + c_1(x+x^2) + c_2(x^2+x^3) + c_3(1+x^3)$$
$$= (c_0+c_3)1 + (c_0+c_1)x + (c_1+c_2)x^2 + (c_2+c_3)x^3$$

implies
$$\begin{cases} c_0 \qquad\qquad + c_3 = b_0 \\ c_0 + c_1 \qquad\qquad = b_1 \\ \quad c_1 + c_2 \qquad = b_2 \\ \qquad c_2 + c_3 = b_3 \end{cases} \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & b_0 \\ 1 & 1 & 0 & 0 & b_1 \\ 0 & 1 & 1 & 0 & b_2 \\ 0 & 0 & 1 & 1 & b_3 \end{array}\right]$$

Ex: $\mathcal{P}_{fin}(\mathbb{R}) = \{\text{polynomials w/ real coeffs}\}$

has basis $\{1, x, x^2, x^3, \dots\}$ which is infinite!

NB: "Nice" parameterizations of spaces yield/use bases..

Ex: Find basis of the vs. of solutions to $\begin{cases} x + y + z = 0 \\ \quad y - 2z = 0 \\ x \quad + 3z = 0 \end{cases}$

Sol: We parameterize the solution space:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{cases} x \quad + 3z = 0 \\ \quad y - 2z = 0 \end{cases}$$

$\therefore$ we have parameterized solutions $\begin{cases} x = -3t \\ y = 2t \\ z = t \end{cases}$

So the solution space is $\left\{ t \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$

So we have basis of solutions $\left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\}$ ☑

Ex: Compute a basis of $\left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} : a+b-c=0 \right\} = V$.

Sol: $\begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \in V \iff \begin{bmatrix} a & b \\ a+b & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ $\quad a = c-b$

So $\begin{bmatrix} a & b \\ a+b & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

So $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis.

$$\begin{bmatrix} c-b & b \\ c & 0 \end{bmatrix} = c\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ is also a basis... ☒

Prop: Let V be a vector space and $B \subseteq V$. The following are equivalent.

1. B is a basis
2. B is both linearly independent and spanning
3. ☆ Every vector in V has a unique expression as a linear combination of vectors from B.
4. B is a maximal linearly independent set.
5. B is a minimal spanning set.